

Hom complexes of Double mapping cylinders of graphs

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Abstract

Let T be a simple graph not containing P_3 as an induced subgraph. We show that $\text{Hom}(T, -)$ maps double mapping cylinders in graphs to homotopy pushouts in topological spaces.

1 Introduction

Let \mathcal{G} denote the category whose objects are finite undirected graphs (no multiple edges) and morphisms are graph homomorphisms taking vertices to vertices and edges to edges. For graphs G and H in \mathcal{G} , the hom complex $\text{Hom}(G, H)$ has been studied extensively for past few years. The hom complex of graphs was defined by Lovász and he conjectured that the connectivity of the hom complex $\text{Hom}(T, G)$ gives a lower bound on the chromatic number of G . He showed [11] that K_2 is a test graph, that is,

$$\chi(G) \geq \chi(K_2) + \text{conn}(\text{Hom}(K_2, G)) + 1, \text{ for all } G.$$

and conjectured that every graph T is a test graph,. It has been since shown that not every graph is a test graph [9]. However, there are several graphs which are indeed test graphs. The class of complete graphs [2], odd cycles [1, 3], bipartite graphs [12] and a class of stable Kneser graphs [15] all are examples of test graphs. Dochtermann and Schultz [8] constructed a class of test graphs T such that the connectivity of $\text{Hom}(T, G)$ gives a tight lower bound on the chromatic number of G .

In this context, one of the main problems is to understand the topology of the hom complex of a pair of graphs. Csorba and Lutz [4] showed that $\text{Hom}(C_5, K_n)$ is a PL-manifold. Čukić and Kozlov studied the homotopy type of $\text{Hom}(C_k, C_n)$ in [6] and $\text{Hom}(G, K_n)$ in [5]. C.Schultz [14] studied the topology of $\text{Hom}(C_k, K_n)$ to give a proof of the graph colouring theorem. Schultz [13] related the topology of $\text{Hom}(K_2, G)$ with the topology of $\text{Hom}(C_{2r+1}, G)$. In general, computing the hom complex for arbitrary graphs G and H is often tedious and not much is known about the topology of $\text{Hom}(G, H)$ for arbitrary graphs G and H .

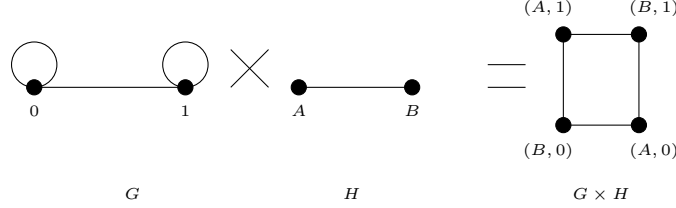
In this article, we study the topology of Hom complexes by breaking it down to smaller and possibly computationally easier peices. Let \mathcal{G}' denote the subcategory of \mathcal{G} of graphs G such that P_3 is not an induced subgraph of G . In this article, we prove (Theorem 3.10) that if T is any graph of \mathcal{G}' then, the functor $\text{Hom}(T, -) : \mathcal{G} \rightarrow \text{Top}$ maps double mapping cylinders in graphs to homotopy pushouts in topological spaces. The subcategory \mathcal{G}' contains all complete graphs K_n , chordal graphs, complete bipartite graphs $K_{m,n}$. If G can be written as a double mapping cylinder of some graphs then, $T \in \mathcal{G}'$, $\text{Hom}(T, G)$ can be computed by computing hom complexes $\text{Hom}(T, H)$ for appropriate subgraphs H of G .

We begin by defining double mapping cylinder in the category of graphs. We then make a few observations about the polytopes in $\text{Hom}(T, G)$ for graphs in \mathcal{G} . In section 3, we prove our main theorem and compute the homotopy type of some Hom complexes using Theorem 3.10.

2 Double mapping cylinders in \mathcal{G}

The category \mathcal{G} has finite products and coproducts. It is easy to verify that disjoint union of graphs is the coproduct in the category of graphs. The product graph is given as follows.

Definition 2.1. *Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ in \mathcal{G} , their cartesian product is defined to be the graph $G \times H = (V(G \times H), E(G \times H))$ where $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) \subset (V(G) \times V(H)) \times (V(G) \times V(H))$ such that $\{(g_1, h_1), (g_2, h_2)\} \in E(G \times H) \Leftrightarrow \{g_1, g_2\} \in E(G)$ and $\{h_1, h_2\} \in E(H)$. For example:*



It is easy to verify that this defines the product in the category of graphs. Dochtermann [7] defines the notion of \times -homotopy of graphs analogously to homotopy of topological spaces and it is easy to see that the chromatic number is a \times -homotopy invariant. In fact, note that if there exist graph homomorphisms $A \rightarrow B$ and $B \rightarrow A$ then the chromatic number of these two graphs should be the same. Clearly it is not a complete invariant.

Let I_n denote the graph with vertex set, $V(I_n) = \{0, 1, \dots, n\}$, and edge set, $E(I_n) = \{\{i, i\}, \{i, i+1\}, \{n, n\} \mid i = 0, \dots, n-1\}$.

Definition 2.2. *Two graph homomorphisms $f, g : G \rightarrow H$ are called \times -homotopic if there exists a natural number n and a graph homomorphism $F : G \times I_n \rightarrow H$ such that $F(a, 0) = f(a)$ and $F(a, n) = g(a)$. Denoted $F : f \simeq_{\times} g$.*

Definition 2.3. *A graph homomorphism $f : G \rightarrow H$ is called a \times -homotopy equivalence if there exists a graph homomorphism $g : H \rightarrow G$ such that $gf \simeq_{\times} 1_G$ and $fg \simeq_{\times} 1_H$. If there exists a \times -homotopy equivalence $G \rightarrow H$ then G and H are said to be \times -homotopy equivalent.*

Let A, B and C be graphs and $f : A \rightarrow B$ and $g : A \rightarrow C$ be graph homomorphisms. We define the double mapping cylinder as $D_n = B \sqcup_f (A \times I_n) \sqcup_g C = (B \sqcup (A \times I_n) \sqcup C) / \sim$, where \sim denotes the equivalence relation $f(a) \sim (a, n)$ and $g(a) \sim (a, 0)$, and $\{[x], [y]\} \in E(D)$ if there exists some $x_0 \in [x]$, $y_0 \in [y]$ such that $\{x_0, y_0\} \in E(B) \cup E(A \times I_n) \cup E(C)$. By abuse of notation, we will simply write x to denote its equivalence class $[x]$.

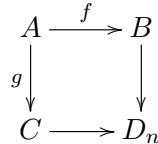


Figure 1: Double mapping cylinder.

Consider the following commutative \times -homotopic diagram for a fixed n .

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 g \downarrow & \searrow & \downarrow & \searrow & \\
 C & & A' & \xrightarrow{f'} & B' \\
 & \searrow & g' \downarrow & & \\
 & & C' & &
 \end{array}
 ,$$

Then there is a \times -homotopy between D_n and D'_n . For $n < m$, D_n is not \times -homotopic to D_m since there does not exist any graph homomorphism from D_n to D_m . Thus D_n depends on n .

Definition 2.4. For any two arbitrary graphs T and G , *Hom complex*, $\text{Hom}(T, G)$ is the polyhedral complex defined by the condition: $\eta = \prod_{x \in V(T)} \eta_x \in \text{Hom}(T, G)$ if and only if for any $x, y \in V(T)$, if $\{x, y\} \in E(T)$ then $\eta(x) \times \eta(y) \subset E(G)$ is a complete bipartite subgraph of G , where $\eta : V(T) \rightarrow 2^{V(G)} \setminus \{\emptyset\}$ and η_x denotes the simplex whose set of vertices is $\eta(x)$.

For a subcategory \mathcal{G}' of \mathcal{G} and $T \in \mathcal{G}'$, we show that $\text{Hom}(T, -)$ functor takes D_n to homotopy pushout of the diagram

$$\begin{array}{ccc}
 \text{Hom}(T, A) & \xrightarrow{f_T} & \text{Hom}(T, B) \\
 g_T \downarrow & & \\
 \text{Hom}(T, C) & &
 \end{array}$$

The theorem implies that $\text{Hom}(T, D_m) \rightarrow \text{Hom}(T, D_n)$ is a weak equivalence for all T in \mathcal{G}' . In this context, the notion of double mapping cylinder is independent of n .

Lemma 2.5. For any two graphs $G, H \in \mathcal{G}$, the spaces $\text{Hom}(G, H) \times I$ and $\text{Hom}(G, H \times I_n)$ are homotopy equivalent.

Proof. We observe that the interval I is contractible. Therefore, $\text{Hom}(G, H) \times I$ is homotopy equivalent to $\text{Hom}(G, H)$. Since $H \times I_n$ can be folded to H , therefore $\text{Hom}(G, H \times I_n)$ is homotopy equivalent [10] to $\text{Hom}(G, H)$. Thus $\text{Hom}(G, H) \times I$ is homotopy equivalent to $\text{Hom}(G, H \times I_n)$. \square

Definition 2.6. Let $T \in \mathcal{G}$ be a graph. A polytope of $\text{Hom}(T, D_n)$ that belongs to either $\text{Hom}(T, A \times I_n)$, $\text{Hom}(T, B)$ or $\text{Hom}(T, C)$ is defined as a pure polytope. Polytopes which are not pure are defined to be mixed polytopes.

Definition 2.7. A polytope of $\text{Hom}(T, D_n)$ is said to be a maximal polytope if it is not a proper face of any other polytope.

Let T be any connected graph. Let $\eta \in \text{Hom}(T, D_n)$ be any maximal polytope, where $D_n = B \sqcup_f (A \times I_n) \sqcup_g C$ as defined above. Then η is one of the following forms:

1. $\text{Im}(\eta) \subset f(A) \cup (V(A) \times \{1, \dots, n-1\}) \cup g(A)$,
2. $\text{Im}(\eta) \subset V(B)$,
3. $\text{Im}(\eta) \subset V(C)$,
4. $\text{Im}(\eta) \cap V(B) \neq \emptyset$ and $\text{Im}(\eta) \cap (V(A) \times \{0, \dots, n-1\}) \neq \emptyset$,

5. $Im(\eta) \cap V(C) \neq \emptyset$ and $Im(\eta) \cap (V(A) \times \{1, \dots, n\}) \neq \emptyset$, or
6. $Im(\eta) \cap V(B) \neq \emptyset$, $Im(\eta) \cap (V(A) \times \{1, \dots, n-1\}) \neq \emptyset$ and $Im(\eta) \cap V(C) \neq \emptyset$.

Therefore, polytopes of the form (1)-(3) are pure, and of the form (4)-(6) are mixed.

Denote $X = V(B) \setminus V(f(A))$, $Y = V(f(A))$, $Z = V(A) \times \{n-1\}$ and $Z' = V(D_n) \setminus (X \cup Y \cup Z)$.

We give a representative picture of all these sets in the following figure.



Lemma 2.8. *Let $T \in \mathcal{G}$ be a graph without any isolated vertex. Let $\eta \in \text{Hom}(T, D_n)$ be any maximal polytope, where $D_n = B \sqcup_f (A \times I_n) \sqcup_g C$. Let $\eta \in \text{Hom}(T, D_n)$ then by definition, $\eta = \prod_{t \in V(T)} \eta(t)$, where each $\eta(t)$ is a simplex.*

1. *If $\eta(t) \cap (X \cup Y) \neq \emptyset$ for every $t \in V(T)$ then, $Im(\eta) \subset X \cup Y \cup Z$.*
2. *Let $t \in V(T)$, if $\eta(t)$ intersects with both X and Z then, $\eta(t)$ intersects with Y as well.*
3. *If $\eta(t) \cap (X \cup Y) \neq \emptyset$ for every $t \in V(T)$ then, $Im(\eta(t)) \cap V(C) = \emptyset$.*

Proof. 1. Let $t \in V(T)$. Since T does not have any isolated vertex, t has non-empty neighbourhood. Let $\{t, t'\} \in E(T)$ and $x \in \eta(t')$. Then the condition $\eta(t) \times \eta(t') \subset E(D_n)$ implies that $x \in X \cup Y$.

2. Take $y \in \eta(t) \cap X$, $z = (a, n-1) \in \eta(t) \cap Z$. Let $t' \in N_T(t)$ and take $x \in \eta(t')$. Then, the condition $\eta(t) \times \eta(t') \subset E(D_n)$ implies that $x \in Y$, since only vertices that are adjacent to elements in both $V(B) \setminus f(A)$ and $V(A) \times \{n-1\}$ belongs to $V(A) \times \{n\}$ or equivalently $f(A)$. By assumption $\in E(D_n)$, in particular $\{x, z\} \in E(A \times I_n)$ and therefore, for $z' = (a, n)$, $\{x, z'\} \in E(A \times I_n)$. Since choice of $x \in \eta(t')$ was arbitrary, we have that each element of $\eta(t')$ is adjacent to $z' \in Y$. Since η is a maximal polytope, we have that $z' \in \eta(t)$.

3. Let $\eta \in \text{Hom}(T, D_n)$ be such that $\eta(t)$ intersects with $V(B) = X \cup Y$ for all $t \in V(T)$. Then, by point (1), $Im(\eta)$ does not intersect with Z' . So no such polytope can have pure faces coming from both $\text{Hom}(T, B)$ and $\text{Hom}(T, C)$. Similar is the argument when η has a pure face $\tau \in \text{Hom}(T, C)$.

□

3 Proof of Theorem 3.10

Definition 3.1. A graph G is said to be connected if for any two vertices x, y of G , there exists a path of edges joining x and y .

Definition 3.2. Let P_n denote the path graph on $n + 1$ vertices. Note that $P_1 = K_2$.

Definition 3.3. A subgraph H of graph G is a graph such that $V(H) \subset V(G)$ and $\{x, y\} \in E(H) \implies \{x, y\} \in E(G)$. An induced subgraph H of a graph G is a subgraph of G such that if, $x, y \in V(H)$ and $\{x, y\} \in E(G) \implies \{x, y\} \in E(H)$.

Proposition 3.4. Let T be a connected graph such that P_3 is not an induced subgraphs. Let $\sigma, \tau \in \text{Hom}(T, D_n)$ be two mixed maximal polytopes with pure faces such that $\sigma \cap \tau \neq \emptyset$. Let $\eta \subset \sigma \cap \tau$ then there exists a polytope $\eta' \in \text{Hom}(T, D_n)$ such that η' has a pure face and $\eta' \subset (\sigma \cap \tau)$.

Proof. Note that vertices in $V(A) \times \{n\}$ are identified with vertices in $f(V(A)) \subset V(B)$ in D_n , that is, $[(a, n)] = [f(a)] \in V(D_n)$. Therefore, if $[(a, n)] \in \eta(t)$ for a polytope $\eta \in \text{Hom}(T, D_n)$ then, $\eta(t) \cap V(B) \neq \emptyset$ and $\eta(t) \cap V(A \times I_n) \neq \emptyset$. We have the following cases depending on which pure polytope(s) is contained in σ and τ .

Case 1: Let $\sigma \cap \text{Hom}(T, B) \neq \emptyset$ and $\tau \cap \text{Hom}(T, B) \neq \emptyset$.

Then for every $t \in V(T)$, $\sigma(t) \cap V(B) \neq \emptyset$. By point (1) of Lemma 2.8, $\sigma(t) \subset V(B) \cup (V(A) \times \{n-1\})$ for each $t \in V(T)$. Let $\eta \subset \sigma \cap \tau$, then $\eta \subset V(B) \cup (V(A) \times \{n-1\})$. If $\eta(t) \cap V(B) \neq \emptyset$ for every $t \in V(T)$ then we are done. So assume that there exists some $t' \in V(T)$ such that $\eta(t') \cap V(B) = \emptyset$ that is to say that $\eta(t') \subset V(A) \times \{n-1\}$. Fix a $t \in N_D(t')$. It is easy to see that $(\sigma(t) \cup \tau(t)) \times (\eta(t') \cup \{(a, n) | (a, n-1) \in \eta(t')\}) \subset E(D_n)$.

Consider the polytope $\xi \in \text{Hom}(T, D_n)$ defined by $\xi(t) = \eta(t)$ for all $t \neq t'$ and $\xi(t') = \eta(t') \cup \{(a, n) | (a, n-1) \in \eta(t')\}$. Then maximality of σ and τ implies that $\xi \subset \sigma \cap \tau$. If $\xi(t) \cap V(B) \neq \emptyset$ for every $t \in V(T)$ then, we are done. If not, repeat the same procedure as above.

Since the graph is finite, the above process terminates after finitely many steps to yield a polytope η' . By construction, $\eta \subset \eta' \subset \sigma \cap \tau$ and $\eta' \cap \text{Hom}(T, B) \neq \emptyset$.

A similar argument works if $\sigma \cap \text{Hom}(T, C) \neq \emptyset$ and $\tau \cap \text{Hom}(T, C) \neq \emptyset$.

Case 2: Let $\sigma \cap \text{Hom}(T, A \times I_n) \neq \emptyset$ and $\tau \cap \text{Hom}(T, A \times I_n) \neq \emptyset$.

Let us assume σ and τ do not have any pure faces from $\text{Hom}(T, B)$ or $\text{Hom}(T, C)$ else the theorem follows from the previous case. If $\eta \subset \sigma \cap \tau$ is such that $\eta \cap \text{Hom}(T, A \times I_n) \neq \emptyset$ then, we are done. Otherwise, there is some $p \in V(T)$ such that $\eta(p) \cap V(A \times I_n) = \emptyset$ that is, either $\eta(p) \subset V(B \setminus f(A))$ or $\eta(p) \subset V(C \setminus g(A))$. Without loss of any generality, let us assume that $\eta(p) \subset V(B \setminus f(A))$.

Connectedness of T gives a path between any two vertices of T and the property that T does not have P_3 as an induced subgraph implies that there is a path with two edges between p and any $t \in V(T) \setminus N_T(p)$. Since $\sigma(p) \cap V(B) \neq \emptyset$, this gives that $\text{Im}(\sigma) \subset V(B) \cup V(A) \times \{n-1\}$. Similar argument implies that $\text{Im}(\tau) \subset V(B) \cup V(A) \times \{n-1\}$. Now $\eta \cap \text{Hom}(T, B) = \emptyset$ gives that there exists a vertex $r \in V(T)$ such that $\eta(r) \subset V(A) \times \{n-1\}$. Let P be a path of length 2 between p and r with middle vertex q , say.



Pick $(a_r, n-1) \in \eta(r)$, then for every element $x \in N_T(r)$, $\{(a_r, n-1)\} \times \eta(x) \subset E(D_n)$. Then $\{(a_r, n)\} \times \eta(x) \subset E(D_n)$. We now define a new polytope $\eta' : V(T) \rightarrow 2^{V(D_n)}$ as follows

$$\eta'(t) = \begin{cases} (a_r, n) \cup \eta(r) & \text{if } \eta(r) \cap V(B) = \emptyset, \\ \eta(t) & \text{otherwise.} \end{cases}$$

Clearly $\eta' \subset \sigma \cap \tau$ is a polytope in $\text{Hom}(T, D_n)$ and $\eta' \cap \text{Hom}(T, B) \neq \emptyset$. This contradicts our assumption that σ and τ do not have pure faces from B .

Case 3: Let $\sigma \cap \text{Hom}(T, B) \neq \emptyset$ or $\sigma \cap \text{Hom}(T, C) \neq \emptyset$ and $\tau \cap \text{Hom}(T, A \times I_n) \neq \emptyset$.

Without loss of generality, let $\sigma(t) \cap V(B) \neq \emptyset$ for every $t \in T$. Now point (1) of Lemma 2.8 implies that $\sigma(t) \subset V(B) \cup (V(A) \times \{n-1\})$. Thus $\eta(t) \subset V(B) \cup (V(A) \times \{n-1\})$ for every vertex t of $V(T)$. If η has a pure face from B or $A \times I_n$ then there is nothing to prove. Otherwise, there is some $p, r \in V(T)$ such that $\eta(p) \subset V(B \setminus f(A))$ and $\eta(r) \subset V(A) \times \{n-1\}$. Now arguing as in Case 2 above, we get a polytope $\eta' \subset (\sigma \cap \tau)$ such that η' has pure faces from B .

Case 4: Suppose σ has pure face from B and τ from C .

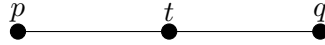
From point (3) of Lemma 2.8, we see that $\sigma(t) \cap V(B) \neq \emptyset$ implies $\sigma(t) \cap V(C) = \emptyset$. Similarly, we get that $\tau(t) \cap V(B) = \emptyset$. Now $\eta \subset (\sigma \cap \tau)$ so $\eta(t) \subset V(A \times I_n)$ for all $t \in V(T)$, and we are done. Put $\eta' = \eta$.

Hence proved. \square

Proposition 3.5. *Let T be a connected graph in \mathcal{G}' . Then every maximal polytope of $\text{Hom}(T, D_n)$ has a pure face.*

Proof. We prove the statement by contradiction. Suppose $\eta \in \text{Hom}(T, D_n)$ is a maximal polytope which does not have a pure face. Then, for some $p, q \in V(T)$ either $\eta(p) \subset V(B) \setminus f(A)$ and $\eta(q) \subset (V(A) \times \{n-1\})$ or $\eta(p) \subset V(C) \setminus g(A)$ and $\eta(q) \subset (V(A) \times \{1\})$. Without any loss of generality, let $\eta(p) \subset V(B) \setminus f(A)$ and $\eta(q) \subset (V(A) \times \{n-1\})$.

The graph T is connected, so there is a path from p to q . Since $\eta(p) \subset V(B) \setminus f(A)$ and $\eta(q) \subset (V(A) \times \{n-1\})$, $\{p, q\} \notin E(T)$ and thus any path between p and q has at least two edges. Since T does not have P_3 as induced subgraph, any shortest path between p and q has at most two edges. Let P be a path from p to q with two edges.



Path P .

Let $x \in N_T(q)$, we claim that $\eta(x) \cap (V(A) \times \{n-2\}) = \emptyset$. Suppose not, then there exists $z \in \eta(x) \cap (V(A) \times \{n-2\})$. Since $T \in \mathcal{G}'$ and $\{p, q\} \notin E(T)$, therefore x is either adjacent to p or t . If x is adjacent to p , then $\eta(p) \subset V(B) \setminus f(A)$ and $\eta(q) \subset (V(A) \times \{n-1\})$ implies that $\eta(x) \subset (V(A) \times \{n\})$. Alternately, if x is adjacent to t , then $\eta(x) \subset (V(A) \times \{n, n-1\})$. Therefore, for every $x \in N_T(q)$, $\eta(x) \subset (V(A) \times \{n\}) \cup (V(A) \times \{n-1\})$, that is $\eta(x) \cap (V(A) \times \{n-2\}) = \emptyset$.

Let $y = (a, n-1) \in \eta(q) \subset (V(A) \times \{n-1\})$ and $z \in \eta(x)$. Then $\{q, x\} \in E(T)$ implies that $(y, z) \in \eta(q) \times \eta(x) \subset E(D_n)$. For $y = (a, n-1) \in \eta(q)$, let y' denote (a, n) then $\{y', z\} \in E(D_n)$. Now define a new polytope

$$\eta'(t) = \begin{cases} \eta(q) \cup \{y' : y \in \eta(q)\} & \text{if } t = q, \\ \eta(t) & \text{otherwise.} \end{cases}$$

Clearly $\eta' \in \text{Hom}(T, D_n)$ and η is a proper face of η' , which contradicts that η is a maximal polytope. Thus every maximal polytope of $\text{Hom}(T, D_n)$ has a pure face. \square

Lemma 3.6. *Let τ be a contractible polyhedral complex and $\delta\tau$ be a closed contractible subcomplex of τ . Then there exists a deformation retract r of τ onto $\delta\tau$.*

Proof. Every polyhedral complex is a CW-complex, therefore $(\tau, \delta\tau)$ is a CW-pair. Then the inclusion $i : \delta\tau \rightarrow \tau$ is a cofibration. Since $\delta\tau$ is contractible, therefore any two maps into $\delta\tau$ are homotopic. Let us consider the homotopy $H : \delta\tau \times I \rightarrow \delta\tau$ between the identity map and the constant map. Then H can be extended to a homotopy $H' : \tau \times I \rightarrow \delta\tau$ because i is a cofibration.

Let $r = H'|_{\tau \times \{0\}}$, then by definition $r|_{\delta\tau \times \{0\}} = H'|_{\delta\tau \times \{0\}} = 1_{\delta\tau}$, that is, r is a retract. Since τ is contractible, $[\tau, \tau]$ has only one element, therefore ir is homotopic to 1_τ . Thus r is a deformation retract. \square

Proposition 3.7. *Let $T \in \mathcal{G}$ be a graph without isolated vertices, and $\eta \in \text{Hom}(T, D_n)$ a maximal polytope, then the union of its pure faces denoted $\delta\eta$, is a contractible subcomplex of η .*

Proof. Let $\eta \in \text{Hom}(T, D_n)$ be a maximal polytope. By point (3) of Lemma 2.8, if $\eta \cap \text{Hom}(T, B) \neq \emptyset$ then $\eta \cap \text{Hom}(T, C) = \emptyset$, therefore η does not have pure faces from both B and C .

If the pure faces of η belong to only one of $\text{Hom}(T, B)$, $\text{Hom}(T, A \times I_n)$ or $\text{Hom}(T, C)$ then $\delta\eta$ is a proper face of η . In particular, $\delta\eta$ is a polytope and polytopes are contractible. Alternately, $\eta \cap \text{Hom}(T, B) \neq \emptyset$ and $\eta \cap \text{Hom}(T, A \times I_n) \neq \emptyset$. If $\eta \cap \text{Hom}(T, B) \neq \emptyset$ then $\eta(t) \subset (V(B) \cup V(A) \times \{n-1\})$ for each $t \in V(T)$ by Lemma 2.8(1). Also, $\eta \cap \text{Hom}(T, A \times I_n) \neq \emptyset$ implies that $\eta(t) \cap V(A \times I_n) \neq \emptyset$ for each $t \in V(T)$. Thus for $t \in V(T)$, $\eta(t) \cap V(A) \times \{n\} \neq \emptyset$ by maximality of η and Lemma 2.8(2).

Let $\sigma_1 = \eta \cap \text{Hom}(T, B)$ and $\sigma_2 = \eta \cap \text{Hom}(T, A \times I_n)$. Now σ_1 and σ_2 both are polytopes and thus contractible. Also,

$$\sigma_1 \cap \sigma_2 = \prod_{t \in V(T)} (\eta(t) \cap (V(A) \times \{n\})) \neq \emptyset.$$

Then from the previous lemma, the polyhedral complex $\sigma_1 \cup \sigma_2$ can be deformation retracted to the shared face $\sigma_1 \cap \sigma_2$.

A similar argument works if $(\eta \cap \text{Hom}(T, C) \neq \emptyset$ and $(\eta \cap \text{Hom}(T, A \times I_n)) \neq \emptyset$. Therefore, $\delta\eta$ is contractible. \square

Lemma 3.8. *Let σ and τ be two intersecting polytopes with contractible subcomplexes $\delta\sigma$ and $\delta\tau$ respectively. If $\delta\sigma \cap \delta\tau$ is contractible then there exists a deformation retract $r : \sigma \cup \tau \rightarrow \delta\sigma \cup \delta\tau$.*

Proof. By definition, for any two polytopes σ and τ , their intersection is either \emptyset or a face of both σ and τ . Given that $\sigma \cap \tau \neq \emptyset$, we get $\sigma \cup \tau$ is contractible.

Since $\delta\sigma \cap \delta\tau$, $\delta\sigma$ and $\delta\tau$ are contractible, $\delta\sigma \cup \delta\tau$ is contractible. Applying Lemma 3.6 to the CW-pair $(\sigma \cup \tau, \delta\sigma \cup \delta\tau)$, we get the desired deformation retract. \square

Theorem 3.9. *Let T be a finite connected graph in \mathcal{G}' . Let $\delta\text{Hom}(T, D_n)$ be the union of all pure polytopes of $\text{Hom}(T, D_n)$. Then there exists a deformation retract $\alpha : \text{Hom}(T, D_n) \rightarrow \delta\text{Hom}(T, D_n)$.*

Proof. Let $\eta, \xi \in \text{Hom}(T, D_n)$ be any two maximal polytopes. Then $\delta\eta$ and $\delta\xi$ are non-empty by Proposition 3.5 and contractible by Proposition 3.7. Suppose $\eta \cap \xi \neq \emptyset$ then, there exists a deformation retract $r : \eta \cup \xi \rightarrow \delta\eta \cup \delta\xi$ by Lemma 3.8.

The space $\text{Hom}(T, D_n)$ has finitely many maximal polytope η_i , where $i = 1, 2, \dots, s$. By induction, it follows that $\cup \eta_i$ deformation retracts to $\cup \delta\eta_i$. Since, $\delta\text{Hom}(T, D_n) = \cup_{i=1}^s \delta\eta_i$, the theorem follows. \square

Theorem 3.10. *Let T be a finite connected graph in \mathcal{G}' . Consider the functor $\text{Hom}(T, -) : \mathcal{G} \rightarrow \text{Top}$ from the category of graphs to topological spaces. Then for the diagram in Figure 1,*

$$\begin{array}{ccc} \text{Hom}(T, A) & \xrightarrow{f_T} & \text{Hom}(T, B) \\ g_T \downarrow & & \downarrow j_1 \\ \text{Hom}(T, C) & \xrightarrow{j_2} & \text{Hom}(T, D_n) \end{array}$$

is the homotopy pushout in Top , the category of topological spaces.

Proof. By Proposition 3.9, $\text{Hom}(T, D_n)$ is homotopy equivalent to $\delta\text{Hom}(T, D_n)$. Now,

$$\delta\text{Hom}(T, D_n) = \text{Hom}(T, B) \sqcup_{f_T} \text{Hom}(T, A \times I_n) \sqcup_{g_T} \text{Hom}(T, C).$$

Lemma 2.5 implies

$$\text{Hom}(T, A \times I_n) \simeq \text{Hom}(T, A) \times I,$$

where I denotes the unit interval $[0, 1]$. Therefore,

$$\delta\text{Hom}(T, D_n) \simeq \text{Hom}(T, B) \sqcup_{f_T} (\text{Hom}(T, A) \times I) \sqcup_{g_T} \text{Hom}(T, C),$$

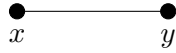
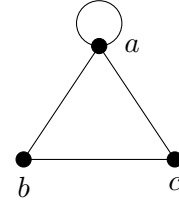
Hence,

$$\text{Hom}(T, D_n) \simeq \text{Hom}(T, B) \sqcup_{f_T} (\text{Hom}(T, A) \times I) \sqcup_{g_T} \text{Hom}(T, C).$$

□

We now apply Theorem 3.10 to compute homotopy type of some Hom complexes.

Example 1: Consider the following graph D on the right. Let $T = C_4$, then it is clear that $T \in \mathcal{G}'$. So to compute $\text{Hom}(C_4, D)$, we first observe that $D = B \sqcup_f (A \times I_1) \sqcup_g C$ for the following as A, B and C :



Graph A



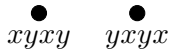
Graph B



Graph C

Let $f : A \rightarrow B$ map x and y to a and $g : A \rightarrow C$ maps x to b , y to c .

Easy computations yield



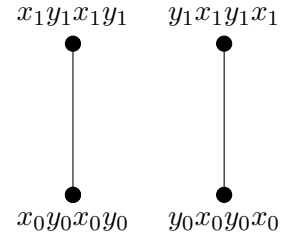
$\text{Hom}(C_4, A)$



$\text{Hom}(C_4, B)$

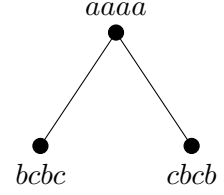


$\text{Hom}(C_4, C)$

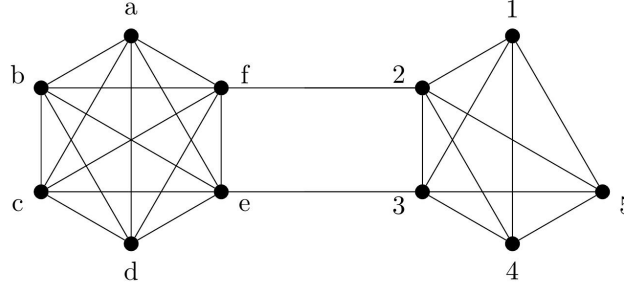


$\text{Hom}(C_4, A \times I_1)$

Theorem 3.10 says that $\text{Hom}(C_4, D) \simeq \text{Hom}(C_4, B) \sqcup_{f_T} \text{Hom}(C_4, A) \times I \sqcup_{g_T} \text{Hom}(C_4, C)$. Therefore, $\text{Hom}(C_4, D)$ is homotopy equivalent to the following topological space which in turn is a contractible space.



Example 2: Let us consider the graph $D = B \sqcup_f (A \times I_1) \sqcup_g C$ with $B = K_p$, $A = K_2$, $C = K_r$; $p, r > 2$ and $f : A \rightarrow B$, $g : A \rightarrow C$ be inclusions. For $p = 6$ and $r = 5$, D looks like:



By Theorem 3.10,

$$\text{Hom}(T, D) = \text{Hom}(T, K_p \sqcup_f K_2 \sqcup_g K_r) \simeq \text{Hom}(T, B) \sqcup_{f_T} (\text{Hom}(T, A) \times I) \sqcup_{g_T} \text{Hom}(T, C).$$

Since f and g are inclusions, the induced maps $f_T : \text{Hom}(T, A) \rightarrow \text{Hom}(T, B)$ and $g_T : \text{Hom}(T, A) \rightarrow \text{Hom}(T, C)$ are also inclusions. Let $T = K_2$, then $T \in \mathcal{G}'$.

Babson and Kozlov [2] proved that $\text{Hom}(K_2, K_n)$ is homotopy equivalent to a $(n - 2)$ -sphere S^{n-2} . So, $\text{Hom}(K_2, K_2) \times I$ is homotopy equivalent to two disjoint line segments, say L_1 and L_2 . Suppose L_1 and L_2 have initial points l_1, l_2 and final points l'_1, l'_2 respectively. Thus, f_T maps l_1, l_2 to $\text{Hom}(T, B) \simeq S^{p-2}$ and g_T maps l'_1, l'_2 to $\text{Hom}(T, C) \simeq S^{r-2}$.

Therefore, $\text{Hom}(K_2, D) \simeq S^{p-2} \vee S^1 \vee S^{r-2}$. One can compute and see that if $p = 6$ and $r = 5$ then, $\text{Hom}(K_2, D)$ is a polyhedral complex with 972 vertices and 15280 facets.

Connectivity of $\text{Hom}(T, D_n)$: We can use Theorem 3.10 to relate the connectivity of the hom complex $\text{Hom}(T, D_n)$ to the connectivity of $\text{Hom}(T, A)$, $\text{Hom}(T, B)$ and $\text{Hom}(T, C)$.

Let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be continuous maps between topological spaces. Let M_f and M_g denote the mapping cylinders of f and g respectively and let W denote the double mapping cylinder. Then $(W; M_f, M_g)$ is an excisive triad and the following sequence is exact.

$$\cdots \longrightarrow H_n(M_f) \oplus H_n(M_g) \longrightarrow H_n(W) \xrightarrow{h} H_{n-1}(X) \longrightarrow H_{n-1}(M_f) \oplus H_{n-1}(M_g) \longrightarrow \cdots$$

where h is the connecting homomorphism from Mayer-Vietoris.

Let $T \in \mathcal{G}'$ and D_n be as in Figure 1. Let us denote the induced maps by f and g again. Let $X = \text{Hom}(T, A)$, $Y = \text{Hom}(T, B)$, $Z = \text{Hom}(T, C)$ and $W = \text{Hom}(T, D_n)$. Since Y is a deformation retract of M_f , $H_*(Y)$ is isomorphic to $H_*(M_f)$. Similarly, $H_*(Z)$ is isomorphic to $H_*(M_g)$.

Then we have the following long exact sequence:

$$\cdots \longrightarrow H_n(\text{Hom}(T, B)) \oplus H_n(\text{Hom}(T, C)) \longrightarrow H_n(\text{Hom}(T, D_n)) \longrightarrow H_{n-1}(\text{Hom}(T, A)) \longrightarrow \cdots$$

Therefore, the topological connectivity of hom complex $\text{Hom}(T, D_n)$ depends on the connectivity of its pieces $\text{Hom}(T, A)$, $\text{Hom}(T, B)$ and $\text{Hom}(T, C)$.

It is then easy to show that if topological connectivities of $\text{Hom}(T, A)$, $\text{Hom}(T, B)$ and $\text{Hom}(T, C)$ are p , q and r respectively then, for a finite connected $T \in \mathcal{G}'$, $\text{connHom}(T, D_n) \geq \min\{p - 1, q, r\}$.

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